A Whittle Index approach to Minimizing Functions of AoI

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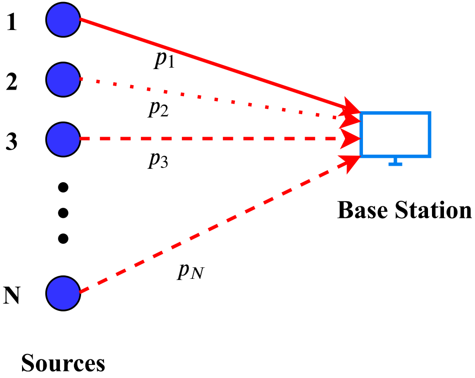
Introduction

- Multiple active sources sending status updates to a monitoring station over a wireless network



- Timely delivery of updates needed to ensure fresh information at the base station

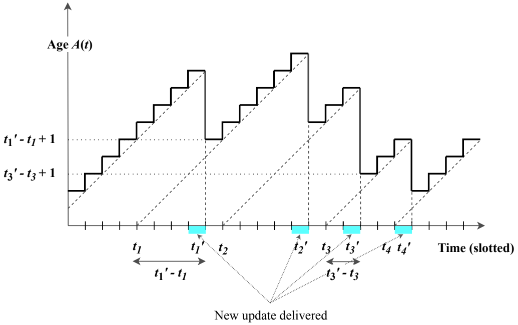
Motivation

- Examples -
  - collecting information for IoT applications
  - aggregating mobility data in vehicular networks
  - realtime surveillance and monitoring
  - networked control systems
- Having the freshest available data is essential to system performance
- AoI - metric that characterizes freshness of information
- Low AoI $\implies$ fresh information $\implies$ better performance

What is Age of Information?

Age of Information $A(t)$ at the destination increases linearly with time, unless it receives a useful update about the source. When an update is received age drops to the time since the update was generated.

$$A(t+1) = \begin{cases} A(t)+1, & \text{if no update at time } t \\ t+1-t_i, & \text{if update } i \text{ is delivered.} \end{cases}$$



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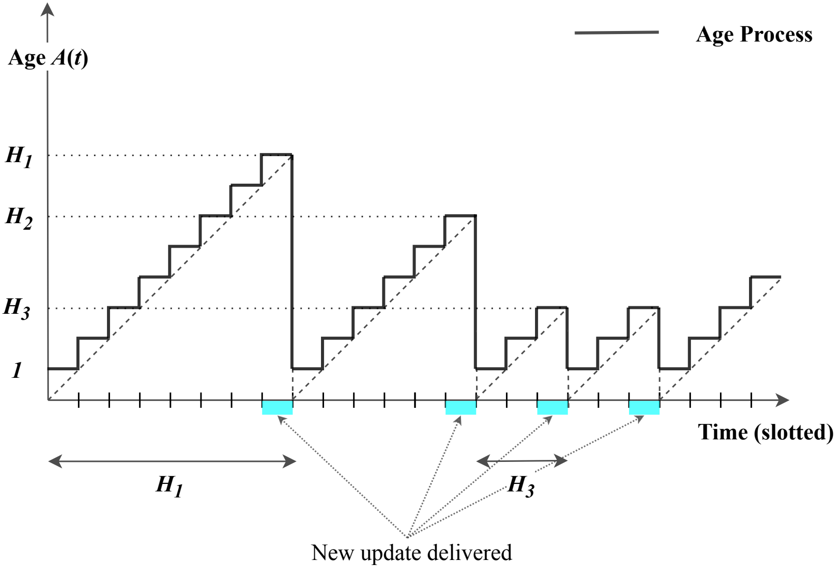
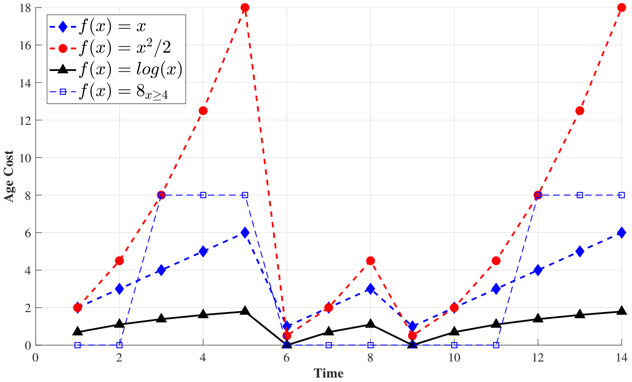
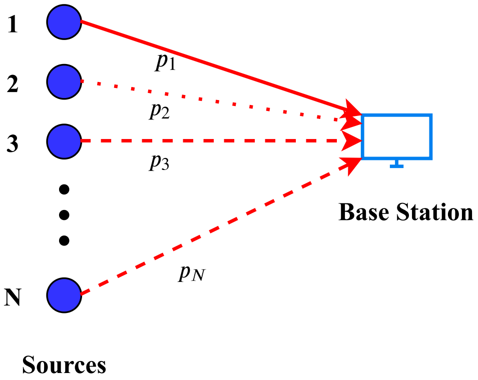


Functions of AoI

- We consider general **non-decreasing cost functions** of AoI for minimization, $f_i(\cdot)$ for source i



Setup



- Set of active sources and a monitoring base station
- Sending updates over a broadcast wireless network
- Slotted system - only one update at a time
- Channel connectivity is i.i.d. Bernoulli

Goal

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- Let π be a scheduling scheme. The expected average cost of age for source i, under policy π is given by

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- Our goal is find a causal policy π that minimizes the sum of average costs of age of sources

$$C^* = \min_{\pi\in\Pi} \sum_{i=1}^{N} C_i^{\text{ave}}(\pi).$$

Functions of Age

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- Optimal policy found to be periodic, **requires high complexity**
- We provide a **low complexity policy** that has near optimal performance in practice for **both reliable and unreliable channels**

Networked Control Systems

- Functions of AoI also motivated by networked control systems in [Champati, et al. 2019] and [Klugel, et al. 2019]
- *Setup* - Estimating an LTI system over an unreliable wireless link with communication cost

Networked Control Systems

- Functions of AoI also motivated by networked control systems in [Champati, et al. 2019] and [Klugel, et al. 2019]
- *Setup* - Estimating an LTI system over an unreliable wireless link with communication cost
- Minimizing the sum of estimation error and transmission cost = Minimizing a non-decreasing age-cost function
- Controlling multiple systems requires solving functions of age scheduling

Linear AoI

- Minimization of weighted sums of AoI considered in multiple settings starting with [Kadota, et al. 2016]
- Low complexity constant factor optimal policies in this setting
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- Results rely on finding a stationary randomized policy that is factor-2 optimal
- We show that stationary randomized policies can be **infinitely worse** than the optimal policy for non-linear functions of AoI

The Restless Multi-Armed Bandit Problem

A Markov Bandit is characterized by an MDP which is:

- either **OFF** - frozen at its current state and gives no reward;
- or **ON** - evolves according to a Markov chain P and has a reward function $f(\cdot)$.

In contrast, when a bandit is **restless**, it has two modes:

- **OFF** - continues the process according to Markov chain P_0 and has a reward function $f_0(\cdot)$;
- **ON** - continues the process according to Markov chain P_1 and has a reward function $f_1(\cdot)$;

RMAB consists of multiple such MDPs, out which only a few can be turned **ON** at any given time.

f-AoI Minimization as RMAB

We formulate the functions of AoI minimization as a RMAB problem. If source i transmits in time-slot t then the state evolution is given by

$$A_i(t+1) = \begin{cases} A_i(t) + 1, & \text{w.p. } 1 - p_i \\ 1, & \text{w.p. } p_i. \end{cases}$$

If the source is not active in time-slot t, then the state evolution is given by

$$A_i(t+1) = A_i(t) + 1.$$

For every source i, the cost function $f_i : \mathbb{Z}^+ \to \mathbb{R}^+$ maps the states(age) of the source to its cost.

Whittle Index Approach

Whittle Index: low complexity heuristic with good performance for RMAB problems. Steps to find a Whittle index -

1. Formulate problem to be solved as a restless bandit problem
2. Formulate the *decoupled problem (dp)*
3. Find optimal policy for *dp* and establish *indexability*
4. Find an expression for the Whittle Index

Note that we have already formulated f-AoI minimization as a RMAB problem.

Decoupled Problem

The next step is to look at the RMAB problem with a single bandit/source and a constant charge for transmission.

Definition

Decoupled Problem

Consider a single source with the cost function $f : \mathbb{Z}^+ \to \mathbb{R}^+$. Let its age be $A(t)$. Its evolution is given by

$$A(t+1) = \begin{cases} A(t)+1, & \text{if not active at time t} \\ 1, & \text{otherwise.} \end{cases}$$

There is a strictly positive activation charge C to be paid in every time-slot that the arm is pulled.

Optimal Threshold Policy

Theorem

The optimal policy for the decoupled problem is a stationary threshold policy. Let H satisfy

$$f(H) \leq \frac{\sum_{j=1}^{H} f(j) + C}{H} \leq f(H+1).$$

Then, the optimal policy is to activate the arm at time-slot t if $A(t) \geq H$ and to let it rest otherwise. If no such H exists, the optimal policy is to never activate the arm.

Indexability

The **indexability property** states that as the activation charge C increases from 0 to ∞, the set of states for which it is optimal to activate the arm decreases monotonically from the entire set $\mathbb{Z}^+$ to the empty set $\{\phi\}$.

- We establish that the indexability property is indeed true for the decoupled problem
- The proof relies on the monotonicity of the age cost function
- The Whittle Index approach states that if a problem is indexable, one can find a *Whittle Index policy* that will have good performance

Whittle Index

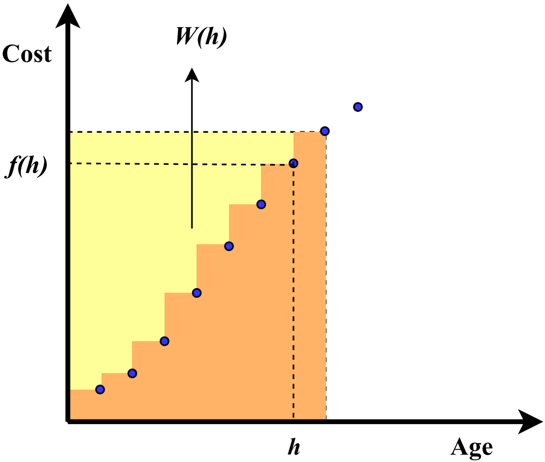
Definition

Whittle Index

Given indexability, $W(h)$ is the infimum charge C that makes both decisions (transmit, not transmit) equally desirable at age h. The expression for $W(h)$ is given by

$$W(h) = hf\left(h+1\right) - \sum_{j=1}^{h} f(j).$$

Whittle Index Interpretation



The yellow area represents the Whittle Index $W(h)$ for age h. It is easy to see that $W(\cdot)$ is monotone, if $f(\cdot)$ is monotone.

Whittle Index Policy

Let $W_i(x) := xf_i\big(x+1\big) - \sum_{j=1}^{x} f_i(j)$ represent the index function for the i^{th} decoupled problem. Then,

Definition

Whittle Index Policy

$\pi^W(t)$ is the action taken by the Whittle Index Policy at time t, given by

$$\pi^W(t) = \arg\max_{1\le i\le N} \Big\{ W_i\big(A_i(t)\big) \Big\}.$$

Is the Whittle Index Optimal?

- Typically NO, just a well performing heuristic
- Optimality is usually established for
  - symmetric systems using a coupling argument or
  - in the limit as system sizes go to infinity, using a fluid limit approach

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- Optimality is usually established for
  - symmetric systems using a coupling argument or
  - in the limit as system sizes go to infinity, using a fluid limit approach
- We establish optimality of the Whittle index for a **finite asymmetric system (**$N = 2$**)**

Theorem

For the functions of age problem with **reliable channels and two sources**, *the Whittle index policy is exactly optimal.*

Unreliable Channels

We establish indexability and derive a similar Whittle Index for the unreliable channels case.

Theorem

The indexability property holds for the decoupled problem. The expression for $W_i(h)$ is given by

$$W_i(h) = p_i^2 h\Big(\sum_{k=1}^{\infty} f_i(k+h)(1-p_i)^{k-1}\Big) - p_i\Big(\sum_{j=1}^{h} f_i(j)\Big).$$

To ensure indexability, we require the following *bounded cost* condition on the age cost functions f_i in addition to monotonicity

$$\sum_{h=1}^{\infty} f_i(h)(1-p_i)^h < \infty.$$

Simulation Results

No. of Sources	Setting	Optimal Cost	Whittle Index Cost
2	A_1 (reliable) A_2 (unreliable)	**21.95** 36.12	**21.95** 36.28
3	D_1 (reliable) D_2 (unreliable)	**44.23** 161.19	**44.23** 161.39
4	E_1 (reliable) E_2 (unreliable)	**73.36** 129.02	**73.36** 130.94
4	F_1 (reliable) F_2 (unreliable)	**87.66** 158.35	**88.27** 159.81

Table: Cost of the Whittle index policy and the optimal policy in different settings.

Conclusions

- We discussed the problem of minimizing functions of age of information over a wireless broadcast network.
- We used a RMAB approach to establish indexability of the problem and found the Whittle index policy.
- For the case with two sources and reliable channels, we were able to show that the Whittle index policy is exactly optimal.
- We also established structural properties of an optimal policy, and a recipe for proving Whittle Index optimality in finite asymmetric settings.

Thank You

Questions?

Supplementary Slides

Whittle Index Optimality

Proof Sketch:

- We define two classes of policies -
  1. strong-switch-type policies
  2. index policies
- We show that these two policy classes are equivalent, for all N
- We establish that there exists a strong-switch-type policy that is optimal for $N \leq 3$
- Finally, we show that the Whittle Index policy is an optimal policy within the class of Index policies, for $N = 2$

Extensions

- Proving constant factor optimality of the Whittle index policy in general, using the structural properties developed in this work
- Considering sources with stochastic arrivals instead of active sources

Strong-Switch-Type Policies

Definition

Strong-switch-type Policies

Consider a stationary policy π that maps every point in the state space ${\mathbb{Z}^+}^N$ to the set of arms $\{1, \ldots, N\}$. We say that such a policy is strong-switch-type if

$$\pi(x_1, \ldots, x_N) = i$$

implies

$$\pi(x'_1, \ldots, x'_N) = i,$$

for all $\boldsymbol{x}$ and $\boldsymbol{x'}$ such that $x'_i \geq x_i$ and $x'_j \leq x_j, \forall j \neq i$.

Structure of Optimal Policies

Jhunjhunwala, et al. had proved that there exists a stationary cyclic policy which is optimal, given reliable channels. We extend this result for our setting with decoupled cost functions

Theorem

For the functions of age problem with reliable channels, no state-action pairs that are a part of the shortest length optimal cyclic policy can violate the strong-switch-type property.

Index Policies and Equivalence

Definition

Index Policies

We say that a stationary policy is an index policy if

$$\pi(x_1, \ldots, x_N) = \arg\max_{1 \le i \le N} \left\{ F_i(x_i) \right\}$$

for all $\boldsymbol{x} \in {\mathbb{Z}^+}^N$, where $F_i : \mathbb{Z}^+ \to \mathbb{R}$ are **monotonically non-decreasing functions** for all i.

Theorem

For the functions of age problem, every policy that is strong-switch-type is also an index policy.

Index Policy Optimality

Theorem

There exists an optimal stationary policy for the functions of age problem with reliable channels and **up to three sources** *that has the strong-switch-type property* **over the entire state-space***.*

Corollary

For the functions of age problem with reliable channels and up to three source, there exists a stationary optimal policy that is an index policy.

Whittle Index Optimality

Finally, we show that among Index policies for two sources, the Whittle Index policy achieves minimum cost. This gives us the following theorem

Theorem

For the functions of age problem with reliable channels and two sources, the **Whittle index policy is exactly optimal**.

Simulation Settings

No. of Sources	Setting	Cost Functions	Connection Probabilities
2	A_1 (reliable) A_2 (unreliable)	$13x_1$, x_2^2 $13x_1$, x_2^2	1, 1 0.9, 0.5
3	D_1 (reliable) D_2 (unreliable)	x_1^2, 3^{x_2}, x_3^4 x_1^2, 3^{x_2}, x_3^4	1, 1, 1 0.66,0.8,0.75
4	E_1 (reliable) E_2 (unreliable)	x_1^3, 2^{x_2}, $15x_3$, x_4^2 x_1^3, 2^{x_2}, $15x_3$, x_4^2	1,1,1,1 0.7,0.9,0.67,0.8
4	F_1 (reliable) F_2 (unreliable)	x_1^3, e^{x_2}, $15x_3$, x_4^2 x_1^3, e^{x_2}, $15x_3$, x_4^2	1,1,1,1 0.8,0.85,0.7,0.66

Table: Cost functions and channel connection probabilities for different settings.